The Negation of Impossibility

Abstract

In 1951 Kenneth Arrow published a book in which he proved that social choice was impossible. There was no way to amalgamate individual preference orderings into a social preference ordering in such a way that certain rational and normative conditions were met. Later Gibbard and Satterthwaite proved that any such amalgamation of individual preference orderings in which there was no advantage to any individual to use strategy to order their preferences insincerely in order to get a better result for themselves was impossible or led to the selection of a dictator. These impossibility theorems have been thought to rule out political direct democracy and also welfare economics giving credibility to the implication that representative democracy and capitalist economics are the best systems that can be devised.

Instead of simple amalgamation, we have devised a more general information processing system which accepts inputs from individual choosers as either preference orderings or utilitarian ratings and outputs a social choice which can be in the form of either cardinal rating or ordinal ranking information. This system is utility based but processes the information in such a way as to alleviate concerns about interpersonal comparisons of utility. It is a hybrid utilitarian approval social choice system. Instead of the individual choosers using strategy, the system itself maximizes the efficacy of each individual input thus disincentivizing individuals from choosing insincerely. It also meets Arrow's five rational and normative conditions thus proving that social choice is not impossible. The result is that a utility based social choice system has been devised which negates both impossibility theorems and should give new life to welfare economics and political direct democracy.

Introduction

In Social Choice and Individual Values, Kenneth Arrow wrote[1] "In a capitalist democracy there are essentially two methods by which social choices can be made: voting, typically used to make 'political' decisions, and the market mechanism, typically used to make 'economic' decisions." Initially, Arrow does not distinguish between political and economic systems claiming that both are means of formulating social decisions based on individual inputs. Arrow then purports to show that there is no rational way to make social decisions based on the amalgamation of individual ones thus ruling out welfare economics or economic democracy and also direct political democracy. The dichotomy between political and economic systems remains with the implication that representative democracy and capitalist economics are the best systems that can be devised. Arrow's result, formerly called the *paradox of voting,* was first discovered by the Marquis de Condorcet[2] in 1785. Condorcet's paradox shows that majority preferences can become intransitive when there are three or more options. Arrow basically mathematized Condorcet's insight.

Gibbard[3] and Satterthwaite[4] concurred with Arrow and proved that any social choice system that was strategy proof was also impossible. Gibbard stated: "An individual manipulates a system of voting if, by misrepresenting his preferences, he secures a result he prefers to the result that would obtain if he expressed his true preferences." Satterthwaite showed that the requirement for choosing procedures (what he called voting procedures) of strategyproofness and Arrow's requirements for social welfare functions are equivalent: a one-to-one correspondence exists between every strategy-proof voting procedure and every social welfare function satisfying Arrow's five requirements. The system discussed in this paper puts individual strategy in the hands of the social choice information processing system itself so that individual choosers are disincentivized from voting insincerely. This also satisfies

Arrow's five conditions as we show in the following simple example and prove later on.

Let's say there are two alternatives and 50 individual choosers. Each individual chooser specifies their input as utilities on a scale which is the real line from –1 to +1. Furthermore, let's say individual 1 has a utility of .8 for alternative A and .2 for alternative B. The strategy involved would lead individual 1 to change his sincere utility rating for alternative A to +1 and, similarly, candidate B to –1. This would maximize the chances that A would win based just on individual 1's choice alone. However, if the information processing system, which accepts inputs from the choosers, does the strategy for them and outputs the choice as +1 for A and –1 for B based on individual 1's sincere choice, then there is no incentive for individual 1 to misrepresent their utility ratings, and they can go ahead and submit their sincere utility ratings as .8 for alternative A and .2 for alternative B. Thus, Gibbard-Sattertwaite has been negated because no chooser has an incentive to misrepresent their sincere utilities. Of course, they could misrepresent their utilities giving A +1 and B –1, but there would be nothing gained from doing so since the system does it for them. For more complex systems, individual choosers will actually tend to diminish their satisfaction with the outcome if they choose insincerely.

A major stumbling block for the development of utilitarian social choice systems regards the issue of interpersonal comparisons. It has been thought that scales which measure the utilities of individuals are incompatible, and that any scale chosen upon which all individuals were supposed to rate their utilities would be arbitrary. Arrow states:[5,6] “If we admit meaning to interpersonal comparisons of utility, then presumably we could order social states according to the sum of utilities of individuals under each, and this is the solution of Jeremy Bentham, accepted by Edgeworth and Marshall.” He also states: “The viewpoint will be taken here that interpersonal comparison of utilities has no meaning and, in fact, that there is no meaning relevant to welfare comparisons in the measurability of individual utility.” Thus, according to Arrow, any individual input must be based on individual preference rankings of the form aPbPc ... meaning a is preferred to b is preferred to c etc. or the notation Arrow uses – aRbR, meaning a is preferred or equal to b etc.

Utilities can be measured on a scale such as the real line from –1 to +1 for example. They can be symbolized as the set $U = \{u_1, u_2, \dots u_i, \dots u_n\}$ corresponding to the alternative set $C = \{c_1, c_2, \dots c_i, \dots c_n\}$, in which there are n alternatives or candidates. In general there will be a utility, u_i, for each possible alternative,c_i, for each individual chooser. If $u_a > u_b$, meaning the utility of alternative a is greater than the utility of alternative b, then aPb and vice versa. A set of utilities will produce both preference ratings and preference orderings. We will show that, for the information processing system considered here, any affine linear transformation of an individual's set of utility ratings will yield the same output or social choice results, and, therefore, it doesn't matter which scale an individual chooses. This is not to say that the utility scale chosen by an individual is not meaningful to themselves, but just that it is meaningless in terms of their contribution to the final output of the system we analyze in this paper. To be clear we are *not* invoking the "one man, one vote" principle.

We develop a social choice system that is utility based, but which overcomes the objections of arbitrariness of utility scales, is strategyproof and also meets Arrow's five normative and rational criteria. Therefore, social choice is not impossible, and the possibility of other such systems exists.

Utilitarian and Approval Choosing

Utilitarian and approval choosing are exactly analogous to utilitarian voting (UV) and approval voting (AV), and, therefore, "voting" and "choosing" are used interchangeably for the purposes of this paper. Also the words "alternative" and "candidate" will be used interchangeably.

Arrow sets up the problem so that each individual chooser orders all alternatives and then society is required to come up with an ordering that is best according to his stated criteria. He states[9] “In the theory of consumer's choice each alternative would be a commodity bundle; ... in welfare economics, each alternative would be a distribution of commodities and labor requirements. … in the theory of elections, the alternatives are candidates.” The method constructed in this paper inputs information from the individual choosers which can be either in the form of ratings or rankings and outputs information in the form of complete social rankings from which social ratings can be derived since individual ratings are known. For example, if alternative A is one of the winners in a multi-winner election, we can compute A's average utility over the whole set of voters since we know how each voter rated A on their individual input. Summing utilities over all winners would give the social utility of a multi-winner election, for example.

Arrow's assumption of input preference orderings or rankings for each individual is a tacit assumption of equal utility scales for each individual equivalent to the “one man, one vote” principle. With the assumption that orderings represent equally spaced utilities, we can convert orderings or rankings to ratings. This may or may not be a very accurate representation of the underlying utilities, but it's the best information available if only orderings are known. For the system considered here and without loss of generality, any scale can be used for this procedure.

In order to negate the Gibbard-Sattertwaite theorems, which maintain that every choosing system for which an individual chooser could use strategy to improve the outcome for themselves violates Arrow's conditions, we choose a social choice processing system which itself implements the optimum strategy for each individual. The system we describe here involves placing a threshold in the utility scale corresponding to an individual chooser's alternative set such that the expected value of utility of the set above threshold is maximized in the social choice outcome. All alternatives with corresponding sincere utilities above threshold are given positive approval style choices which corresponds to raising each sincere utility in that set to the maximum utility strategically. Alternatives with corresponding utilities below threshold are given negative approval style choices which corresponds to lowering each sincere utility in that set to the minimum utility strategically. This represents the optimal strategy which is implemented by the information processing system itself and not the individual chooser. Therefore, there is no incentive for an individual to use strategy or vote insincerely. Even if this is not the optimal strategy for this or any other system, to the extent that an optimal strategy is known by the individual chooser, it is also presumed to be known by the system itself. Therefore, there is no incentive for an individual to use strategy because either it would have no effect on the outcome because the system would agree with the chooser's strategy or it would have a suboptimal effect on the outcome if the system changed the chooser's strategized choice. In the latter case it would only diminish the utility of the outcome for them personally.

Claude Hillinger[8] has made the case for utilitarian voting:

“There is, however, another branch of collective choice theory, namely utilitarian collective choice, that, instead of fiddling with Arrow’s axioms, challenges the very framework within which those axioms are expressed. Arrow’s framework is *ordinal* in the sense that it assumes that only the information provided by individual orderings over the alternatives are relevant for the determination of a social ordering. Utilitarian collective choice assumes that individual preferences are given as *cardinal* numbers; social preference is defined as the sum of these numbers. The fact that voting procedures are cardinal suggests that cardinal rather than ordinal collective choice theory should be relevant.”

The difference between Hillinger's statement and the system considered here is that social preference is *not* defined as the sum of cardinal numbers. There is a transformation from the cardinal inputs to approval style outputs which can then be converted back into cardinal numbers if desired. Hence, the system we examine is a utilitarian approval hybrid.

Hillinger[9] advocates Evaluative Voting (EV) in which the voter assigns a value to each candidate. For example, EV-3 assigns one of the values (–1,0,+1), and then the values are summed over all candidates to determine the winner. The problem with approval voting, which Hillinger claims to ameliorate, is what to do with the candidates that are neither strongly approved of or strongly disapproved of [i.e.] those in the middle. Hillinger assigns these candidates a value of zero. He[10] asserts:

"Another criticism of *AV*, is due to Lawrence Ford, chair of the mathematics department, Idaho State University, ... :

> One big flaw [of *AV*] is that most voters are fairly positive of their favorites and fairly positive of those they hate, but wishy-washy in the middle. If they choose randomly for or against approval in that middle range, the whole election can become random.

Directed against *AV*, this criticism has some validity because under *AV*, not to approve a candidate is equivalent to being against him. This puts the voter in a bind of having to be for or against, when in fact he lacks the relevant information for [such] a judgment."

For our purposes we adopt Hillinger's EV-3 method. The use of an optimal threshold to determine which candidates get an approval style vote of +1 and which get an approval style vote of –1 clears up one of the criticisms of approval voting regarding what to do about candidates that a voter is wishy washy about. All those above threshold get a +1 vote; all those below get a –1 vote. The only ones who would get a 0 vote would be those that fell directly on the threshold.

Aki Lehtinen[11] concludes that Arrow's Impossibility Theorem is not relevant in the final analysis: "Arrow's impossibility result and the closely related theorems given by Gibbard and Satterthwaite are unassailable as deductive proofs. However, we should not be concerned about these results because their most crucial conditions are not justifiable. Fortunately, we know that strategy-proofness is usually violated under all voting rules and that IIA [Independence of Irrelevant Alternatives] does not preclude strategic voting." Contrary to Lehtinen's assertion, strategyproofness is not violated if the system itself applies the strategy instead of the individual choosers.

Optimal Threshold Social Choice

The Optimal Threshold Social Choice (OTSC) Information Processing System can be modeled as follows. In general we place a threshold in the scale which measures the utilities of a set of alternatives for each chooser such that the expected utility of the social choice for the set of alternatives above threshold is maximized.

To state the problem formally, let S be the set of all alternatives (political candidates or work-commodity bundles or a distribution of commodities and labor requirements etc.). This set can be thought of as the nominees. For each individual chooser every alternative is associated with a utility. Let W be the set chosen by society based on individual inputs. $W \subset S$. $|W| < |S|$. We call W the winning set. A threshold is placed in each individual chooser's utility rating scale, which we assume, without loss of generality, to be the real line between –1 and +1, with utilities above threshold being converted to maximum approval style choices (+1) and utilities below threshold being converted to minimum approval style choices (–1). The threshold T is a real number, $(-1 < T < 1)$. The threshold is placed so as to maximize the expected utility of the set W for each individual. In general the thresholds will be different for each individual. Summed over the individual choosers, the alternatives representing the top $|W|$ approval style choices will comprise the winning set. It is assumed that each individual specifies a utilitarian style input which represents their sincere utility ratings over the set S. Later we will show that any utility scale will yield the same results so that the issue of interpersonal comparisons is moot. Therefore, all utility scales can be standardized as the real line between –1 and +1. Also if the only information available is ordinal, it can be converted to utilitarian data before being used as inputs to the OTSC system.

To state the parameters formally for each individual: Let C be the set of all candidates, c_i be a particular candidate with associated utility, u_i. For each individual chooser let U be the set of utilities corresponding to all candidates, U_a be the set of utilities above threshold and U_b be the set of utilities below threshold. Let C_a be the set of candidates above threshold and C_b be the set of candidates below threshold. Let u_a be the sum of utilities above threshold and u_b be the sum of utilities below threshold. Let n_a be the number of candidates above threshold and n_b be the number of candidates below threshold so that $n = n_a + n_b$ = total number of candidates with associated utilities.

Let V_a be a random variable which represents the utility of the winning set for each individual chooser.

Then the OTSC system maximizes

$$\mathbf{E(V_a) = \sum_{i=1}^{n_a} p_i u_i}$$

for each individual.

In the absence of polling or probability information for each alternative, $\mathbf{p_i}$ will be the same for each alternative. For the present we assume that polling information is unknown. The method easily extends to the case in which polling information is available. Therefore,

$$\mathbf{E(Va) = p \sum_{i=1}^{n_a} u_i}$$

for each individual chooser.

Let's identify the above expression with a ball and urn problem containing black and white balls. The white balls represent candidates above threshold and the black balls represent candidates below threshold. We posit a “picker” that picks balls randomly one at a time out of the urn without replacement and places the balls in the winning set.

The mathematics for this are the following:

$$\mathbf{p' = \frac{\binom{n_a}{k}\binom{n - n_a}{m - k}}{\binom{n}{m}}}$$

where **p'** equals the probability of k white balls placed in the winning set out of m picks, without replacement, from a finite population of size n containing exactly n_a white balls, wherein each draw can either produce a white ball or a black ball. Exactly which white ball (associated with a particular candidate) is picked is not known so that we use the average utility of above threshold candidates placed in the winning set for each pick. Therefore,

$$E(Va) = p'[k=1][\frac{1}{n_a}\sum_{i=1}^{n_a} u_i] + p'[k=2][\frac{2}{n_a}\sum_{i=1}^{n_a} u_i] + \ldots + p'[k=s][\frac{s}{n_a}\sum_{i=1}^{n_a} u_i]$$

where s = min{m, n_a}

Therefore,

$$E(V_j) = \sum_{k=1}^{s}\left\{\left[\frac{\binom{n_a}{k}\binom{n-n_a}{m-k}}{\binom{n}{m}}\right]\left[\frac{k}{n_a}\right]\left[\sum_{i=1}^{n_a} u_{ij}\right]\right\}$$

where j indicates a particular chooser.

As an approximation we will use the following expression for p_i:

$$p \approx P[c_a \in W \mid n(W \cap C_a) \geq 1]\, P[n(W \cap C_a) \geq 1]$$

where W = winning set; Ca = set of candidates above threshold; c_a = a particular candidate in the set of candidates above threshold; n(W)= number of elements in winning set. This can be interpreted as the probability that a particular above threshold candidate, c_a, is in the winning set given that one or more above threshold candidates are in the winning set times the probability that one or more above threshold candidates are in the winning set. The probability of the i^{th} above threshold candidate (where $1 \leq i \leq n_a$) being in the winning set given that one or more above threshold candidates are in the winning set is $1/n_a$. The probability of one or more above threshold candidates being in the winning set can again be expressed by the hypergeometric function. We let **p** be the probability that at least one above threshold candidate is selected.

p = 1 – p' (every candidate selected is below threshold)

$$= 1 - \frac{\binom{n_a}{0}\binom{n - n_a}{m - 0}}{\binom{n}{m}}$$

$$= 1 - \frac{\binom{n - n_a}{m}}{\binom{n}{m}}$$

$$= 1 - \frac{(n - m)(n - m - 1)\ldots(n - n_a - m + 1)}{n(n - 1)\ldots(n - n_a + 1)}$$

In general we have

$$p = 1 - \left[1 - \frac{n_a}{n}\right]\left[1 - \frac{n_a}{n - 1}\right]\ldots\left[1 - \frac{n_a}{n - i}\right]\ldots\left[1 - \frac{n_a}{n - m + 1}\right]$$

for $\mathbf{m = |W| < n - n_a - 1}$

This reduces to the following expression:

$\mathbf{p = 1 - [1 - (n_a/n)][1 - n_a/(n-1)]\ldots[1 - n_a/(n-i)]\ldots[1 - n_a/(n-m+1)]}$

The expected value of utility associated with above threshold candidates for a particular individual voter is the following:

$$E(V_j) = \left(\frac{p}{n_a}\right)\sum_{i=1}^{n_a} u_{ij}$$

Therefore,

$\mathbf{E(V_j) = p(u_a/n_a)}$

The OTSC filter does the computations for every possible threshold to determine which threshold is best i.e. which threshold results in the maximum value of expected utility for the winning set. It does this for each chooser. All candidates above threshold will have their choices increased to +1, and those below threshold will be decreased to –1. Alternatives whose utilities fall exactly on or close to the threshold will be set to zero. The results for all alternatives will then be tallied over all choosers. In addition to the individual choice thresholds there is a social choice threshold in the voting results corresponding to the size of the winning set. All alternatives with social choice totals above this threshold will be declared members of the winning set. **Maximizing individual chooser satisfaction or utility has to do with the correct placement of the optimal threshold for each chooser.**

The theory advanced here results in approval style choosing in the sense that individual cardinal or ordinal inputs are converted to approval style choices. Historically, approval voting is geared to selecting one candidate from a single member district. In that case it has been shown that votes should be cast for all candidates who are above average with respect to a voter's cardinal rating scale. Smith[12], proves the following: "Mean-based thresholding is optimal range-voting strategy in the limit of a large number of other voters, each random independent full-range." Range voting is similar to utilitarian voting. Lehtinen[13] has used expected utility maximizing voting behavior to indicate which candidates should be given an approval style vote in single member districts. He agrees with Smith that an approval style vote of +1 should be given to all candidates for whom their utility exceeds the average utility of all candidates. All others would get an AV vote of zero. For single member districts then, the optimal threshold is placed at the mean of the sincere ratings for each individual.

Smith and Lehtinen have shown that for a one winner outcome all ratings greater than the individual's average rating are changed to the maximum rating, and all ratings less than the average are changed to the minimum rating. Since we use maximum and minimum ratings of +1 and −1, respectively, in our analysis, this is equivalent to placing the threshold at zero and adjusting the ratings for every candidate with a utility above that threshold to +1 and adjusting the rating for every candidate with a utility below that threshold to –1. Preference ratings falling right on the threshold can be given a zero choice similar to Hillinger's preferred EV-3 voting method. Finally, the approval style choices for each candidate are summed over all choosers, and the candidate with the most approval style choices is declared the winner.

As the threshold is raised, **p** gets smaller while $\mathbf{u_a/n_a}$ gets larger with m and n constant. Since there are fewer alternatives above threshold, the chances for some of them being in the winning set are smaller. Similarly, as the threshold is lowered, the chances are greater for some of the alternatives above threshold to be in the winning set. We want to determine where to place the threshold so as to maximize the expected utility of those candidates above threshold for the individual chooser under consideration.

An example for m = 1 is shown in Appendix 1. A normalized data set of utilities is assumed.

This example agrees with Smith and Lehtinen when the winning set contains only one winner. In Appendix 2 we compare the results for m = 1 and m = 2. This graph shows that, as the winning set increases in size, everything else remaining the same, the individual chooser is more likely to achieve greater utility from the winning set since more of their highly preferred alternatives are likely to be in it. Therefore, the optimal threshold can be increased. Appendix 3 shows results for larger values of m.

As an example let's assume a party is being thrown for the employees of a large company. Each employee nominates the food and drink items they would like to see available at the party. Let's say there are a total of n items nominated, but the grocery list has to be limited to m items. Optimal Threshold Social Choice can be used to determine which m items are in the winning set based on individual utilities for each item.

A more elaborate example is shown in Appendix 6.

Optimal Threshold Social Choice Meets Arrow's Five Conditions

Arrow's five rational and normative conditions are

1) Unrestricted domain.

2) Positive Association of Individual and Social Values

3) Independence of Irrelevant Alternatives (IIA)

4) Citizens' Sovereignty

5) Non-dictatorship

Since any alternative can be given any rating by each individual chooser, number (1) is satisfied. Number (2) is satisfied because raising an alternative's utility in some individual's utilitarian style input from just under to just above threshold will result in that alternative's receiving one more approval style individual choice in the final summation. This would raise the social choice by one for that alternative potentially putting that alternative in the winning set. Similarly, lowering a candidate's rating in some individual's utility scale could eliminate that alternative from the winning set. Number (4) is satisfied since the OTSC system treats all alternatives and citizens in an equal and neutral manner, and number (5) is satisfied since the winning set is based only on individual inputs in such a way that no individual has any more say over the outcome than any other individual.

As for number (3), IIA, first of all utilitarian style sincere ratings for each candidate are assumed to be independent of each other regardless of the composition of the alternative set. So if an individual rates a candidate .5 on the scale which is the real line between −1 and +1, and then another candidate enters the race, it is assumed that the first candidate will still be rated at .5. A candidate's dropping out or entering the race is assumed not to change an individual's sincere ratings for the other candidates. Now consider the case in which, after the election occurs, a candidate dies or drops out. Arrow states[14]: "Suppose that an election is held, with a certain number of candidates in the field, each individual filing his list of preferences, and then one of the candidates dies. Surely the social choice should be made by taking each of the individual's preference lists, blotting out completely the dead candidate's name, and considering only the orderings of the remaining candidates in going through the procedure of determining a winner." Arrow implies that the voting has already occurred, but the final determination of the winner(s) has not been made. If this were the case, the OTSC Information Processing System could blot out the dead candidate's rating from all of the individual rating scales, recompute all the individual thresholds and recompute the winning set. However, this might have the effect of changing the composition of the winning set since all the recomputed individual thresholds may have changed. However, there is no need to do this since the dead candidate can just be blotted out of the previously computed social results. The individual optimal thresholds do not have to be changed. As long as the individual optimal thresholds do not change, the comparative relationships among candidates in the final social choice will not change either. OTSC will produce identical results for all the other candidates if the death occurs after the election takes place but before the final results are made public as is proven in Appendix 4.

More formally, let C(S) be the social choice before candidate j dies and C'(S) be the social choice after candidate j dies. Let $R_1,...,R_n$ be the sets of individual orderings of the n choosers, corresponding to utility sets $U_1,...,U_n$ before candidate j dies and $R'_1,...,R'_n$ be the sets of orderings corresponding to utility sets U'_1,...,U'n after candidate j dies. After removing candidate j from C(S), C'(S) and C(S) are the same because the individual thresholds are fixed at the values they had before candidate j died, and they produce the same social choice except for “blotting out the dead candidate's name” from the social choice.

As an alternative demonstration that the OTSC system complies with IIA, let's assume that blotting out the dead candidate's name from the final social choice produces a solution which is suboptimal in the sense that recomputing the solution from scratch would produce a different final result. Even assuming that the individual thresholds are sub-optimal, the choosers would have no incentive to cheat or strategize because they have already submitted their ballots under the assumption that the dead candidate was still in the race, and Gibbard-Sattertwaite is still satisfied. Therefore, because of the one to one correspondence between Gibbard-Satterthwaite and Arrow, IIA is still satisfied. Certainly, with this alternative to the proof, the final results represent a possible solution, not impossibility. If some of the dropouts were in the winning set, other candidates might be elevated to the winning set to replace them. This would not violate IIA.

Now consider the case in which a new candidate enters the race after the balloting has occurred but before the election results have been published. The added utility rating for that candidate would be submitted to the OTSC system by each individual chooser after the utilities for the other candidates had presumably already been submitted, and the results had already been computed. The OTSC system would then recompute the individual thresholds including the added candidate's utility rating and the final results recomputed. The individual choosers would not have an incentive to rate the added candidate insincerely on their utility scales knowing that the OTSC system would give them the strategically best outcome based on the complete list of submitted utilities. Therefore, candidate add-ons would not incentivize any individual chooser to choose insincerely. Satterthwaite showed that the requirement for choosing procedures of strategyproofness and Arrow's requirements for social welfare functions are equivalent: a one-to-one correspondence exists between every strategy-proof voting procedure and every social welfare function satisfying Arrow's five requirements. Compliance with IIA is satisfied for add-ons since ratings for two candidates at a time can be uploaded for each individual chooser with thresholds recomputed at each step if necessary or as a final step thus demonstrating that the social choice can be arrived at by pairwise comparisons which Arrow's IIA demands.

Arrow boils down Condition 3 to pairwise comparisons [15]**:** "Knowing the social choices made in pairwise comparisons in turn determines the entire social ordering and therewith the social choice function C(S) for all possible environments." If choosers built up their utilities by pairwise comparisons, the results would remain the same as if they specified all their utility inputs at the same time since sincere individual utility specifications for a particular candidate, it is assumed, do not depend on utility specifications for other candidates, and choosers know that choosing insincerely will give them a suboptimal choice in the final selection.

More formally, let C(S) be the social choice before candidate j is added on to the list of candidates and C'(S) be the social choice after candidate is added on. Let R_1,...,Rn be the sets of individual orderings of the n choosers, corresponding to utility sets U_1,...,Un before candidate j is added on and R'_1,...,R'n be the sets of orderings corresponding to utility sets U'_1,...,U'n after candidate j is added on. The issue is moot because only the results corresponding to C'(S) will be published. Therefore, add-ons or drop-outs will not cause IIA to be violated.

Optimal Threshold Social Choice is Strategyproof

Since the data is processed in an optimal manner for each individual voter by the system itself, giving each voter the optimal strategy, the voters have no incentive to misrepresent their preferences or to choose insincerely. They would either choose sincerely or the OTSC filter would process their input in such a way as to give them a suboptimal result or the same result. A social welfare function (Arrow's term) or a voting procedure (Satterthwaite's term) in which the strategy is inherent in the choosing procedure itself and applies to all choosers leads to a system in which there is no advantage to individuals to misrepresent their preference orderings or ratings. Clearly, Gibbard-Satterthwaite's theorems do not apply. The voters do not have an incentive to vote insincerely and the voting system has not led to a dictator. The strategy has been placed in the processing of the choices rather than in each individual chooser's hands. The choosers themselves are disincentivized from choosing insincerely.

The optimum strategy is to set a threshold in each individual's utilitarian style input which gives every alternative above threshold the maximum "vote" and every alternative below threshold the minimum "vote" in such a way as to maximize the expected value of utility of the social choice for each individual. This effectively turns the utilitarian style inputs into approval style "votes," but the connection with the underlying utilitarian basis of the system is maintained since the original utilities are known and can be used to compute the utility of the social choice for each individual and for society in general.

The Issue of Interpersonal Comparisons is Moot

Arrow dwells on the fact that individual utility scales are not compatible. He compares them with the measurement of temperature which is based on arbitrary units and the arbitrary terminal points of freezing and boiling for the Celsius scale and completely different end points for the Fahrenheit scale.[16] "Even if, for some reason, we should admit the measurability of utility for an individual, there still remains the question of aggregating the individual utilities. At best, it is contended that, for an individual, their utility function is uniquely determined up to a linear transformation; we must still choose one out of the infinite family of indicators to represent the individual, and the values of the aggregate (say a sum) are dependent on how the choice is made for each individual. In general, there seems to be no method intrinsic to utility measurement which will make the choice compatible."

Bonner[17] has discussed cardinal utility as follows: "Cardinal measurement is of little use in adding up social welfare if interpersonal comparisons cannot be made. … The scale and origin of every personal index might be different, and – what is more important – any attempt to convert them to a common basis would be open to criticism." We show that even though the scale and origin of every personal index may be different, the OTSC method can process them in such a way that each individual's input will yield maximal results for them. Regardless of any affine linear transformation of each utility scale, the results for OTSC will be the same so that the individual choosers are free to choose any scale they want.

Let's unpack Bonner's statement. First, we admit the measurability of utility for each individual. Let's say that, in general, utility can be measured as points on the real line where $-\infty < x < +\infty$ and x is a point of the real line. It's up to the individual where to place the points, including the end points, corresponding to the utilities of each candidate in the candidate set consisting of n alternatives, $\{c_1, c_2, \ldots, c_n\}$. It is proven in Appendix 5 that, for the OTSC system in particular, the results will be the same no matter which utility scale each individual chooses. Any affine linear transformation of a chooser's utility scale will yield the same results. There is no need to "choose one out of the infinite family of indicators to represent the individual."[18] Consequently, Arrow's statement that "the values of the aggregate are dependent on how the choice is made for each individual" is not true. However, since any scale chosen by each individual will yield the same results, without loss of generality, we can standardize the choosing process by transforming individual scales to the real line between –1 and +1 before input to the OTSC system.

The OTSC procedure converts an individually specified set of utilities regardless of scale to a set of approval style decisions. The +1s represent the choices *for* alternatives in the alternative set; the –1s represent the choices *against* alternatives in the alternative set. This conversion is done in such a way as to maximize the power of each individual choice. Therefore, the choice made for each individual is "compatible" since it's made using the same rationale. No matter which scale an individual chooses, they have no incentive to misrepresent their true utilities.

Amartya Sen stated in his Nobel lecture [19] "... economists came to be persuaded by arguments presented by Lionel Robbins and others (deeply influenced by "logical positivist" philosophy) that interpersonal comparisons of utility had no scientific basis. 'Every mind is inscrutable to every other mind and no common denominator of feelings is possible.' Thus, the epistemic foundations of utilitarian welfare economics were seen as incurably defective." OTSC has shown that there is a sound epistemic basis for a utility based social choice system. The OTSC system is in fact logical positivist *because* it has a sound scientific basis. Proving that Arrow's and Gibbard-Satterthwaite's impossibility results are invalid for just one system such as OTSC proves that social choice is not impossible potentially for other systems as well.

Preference Rankings Can Be Converted to Ratings and Vice Versa

Preference rankings can be converted to ratings for each individual which are then passed through the same OTSC procedure. Since the only information for rankings is of the form aPbPcPd... or aRbRcRd...), we can choose any utility scale as long as the preference rankings are equally spaced along that scale since that is the only information available. We know that the choice of which scale to use is irrelevant. Let's say we choose the real line between –1 and +1. We let the top ranked candidate be placed at +1 and the lowest ranked candidate be placed at –1. The other candidates then would be equally spaced on the scale. Since an optimal threshold exists, the OTSC information processing system outputs approval style positive choices for those candidates represented by utilities above threshold and negative choices for those candidates represented by utilities below threshold for each individual. As we have shown, any affine linear transformation of an individual's utility scale will not change the results of the OTSC processing system. The outputs are in the form of integers and represent the votes or choices for or against each alternative or candidate. The output information is ordinal and complete over all alternatives. Thus both individual inputs and social choice output can be in the form of rankings if utility information is not available. Furthermore, the average utility of the alternatives in the winning set can be computed for each individual since their input utilities are known and for society as a whole since the output ordinal social rankings can also be converted back into cardinal form. This can be done on an individual or a social basis. Thus the social choice inputs and outputs can be either in the form of rankings which Arrow assumed or in the form of ratings or utilities.

Conclusions

It has been shown that social choice is possible thus disproving both Arrow's and Gibbard-Satterthwaite's impossibility theorems which are in essence mathematical tautologies. We have theoretically negated these impossibility theorems by demonstrating a system, the Optimal Threshold Social Choice (OTSC) system, which accepts Arrow's and Gibbard-Satterthwaite's conditions and yet produces actual possible results. The OTSC system accepts individual utilitarian style preference ratings as inputs and outputs approval style social choice preference rankings. The OTSC system processes the inputs in such a way as to maximize the expected utility of the social choice for each individual chooser. This is done by setting a threshold in the input utilitarian data of each individual chooser and outputing positive approval style choices for those candidates above threshold. Thus the input data is converted into approval style outputs which are then summed over all choosers which produces social choice rankings of the alternatives. Since the OTSC converts the utilitarian style inputs to approval style outputs, OTSC is a utilitarian approval hybrid system. The hybridization resolves two issues: it makes the issue of interpersonal comparisons moot, and it gives each chooser an optimal strategy which, when undertaken by the system itself and not at the individual level, disincentivizes individual choosers from choosing insincerely. Any use of strategy by individual choosers would result in a suboptimal or the same outcome for them.

The issue of interpersonal comparisons is moot because any affine linear transformation of an individual's utility scale will produce the same results when processed by the OTSC system. Finally, if inputs are specified as preference rankings rather than ratings, the rankings can be converted to utility style ratings which can then be processed by the OTSC system. The outputs which are in the form of social rankings can also be converted back to ratings since utility information for each individual chooser is known. Based on the social choice, utilities can be computed for each individual or summed for society as a whole.

Arrow's main conclusion has been known since 1785 from the work of the Marquis de Condorcet, but Arrow attempted to elaborate and recast the paradox of voting as a proof that any kind of rational system which purported to determine the public good instead led to a dictatorship. The American and French revolutions of 1776 and 1789 respectively, although originally expressing their zeal for government by the people, ended up enshrining power in representative government precisely because the writers of their Constitutions did not trust the people. One of the most important theoreticians of the French revolution, the Abbe Sieyes, wrote[20], "In a country that is not a democracy –and France cannot be one – the people, I repeat, can speak or act only through its representatives." David Van Reybrouck writes[21], "The French Revolution, like the American, did not dislodge the aristocracy to replace it with a democracy but rather dislodged a hereditary aristocracy to replace it with an elected aristocracy, *'une aristocratie elective'*, to use Rousseau's term." The impossibility theorems of Arrow and Gibbard-Satterthwaite seem to have driven this point home since they claim that direct democracy and welfare economics are impossible leaving only capitalist economics and representative democracy with a sound epistemic basis. The work presented here proves that direct political and economic democracy do in fact have a sound scientific basis and that social choice is not impossible.

Appendix 1

Example for m = 1

$$p = 1 - \frac{n - n_a}{n}$$

Expected value of utility =

$$E(V_a) = p\left(\frac{u_a}{n_a}\right) = \left(\frac{n_a}{n}\right)\left(\frac{u_a}{n_a}\right) = \frac{u_a}{n}$$

If we place the threshold just under –1, $n_a = n$, p = 1, $u_a = 0$, $E(V_a) = 0$.

If we place the threshold just under +1,

$n_a = 1$, $u_a = 1$, p = 1/9 and $E(V_a)=1/9$.

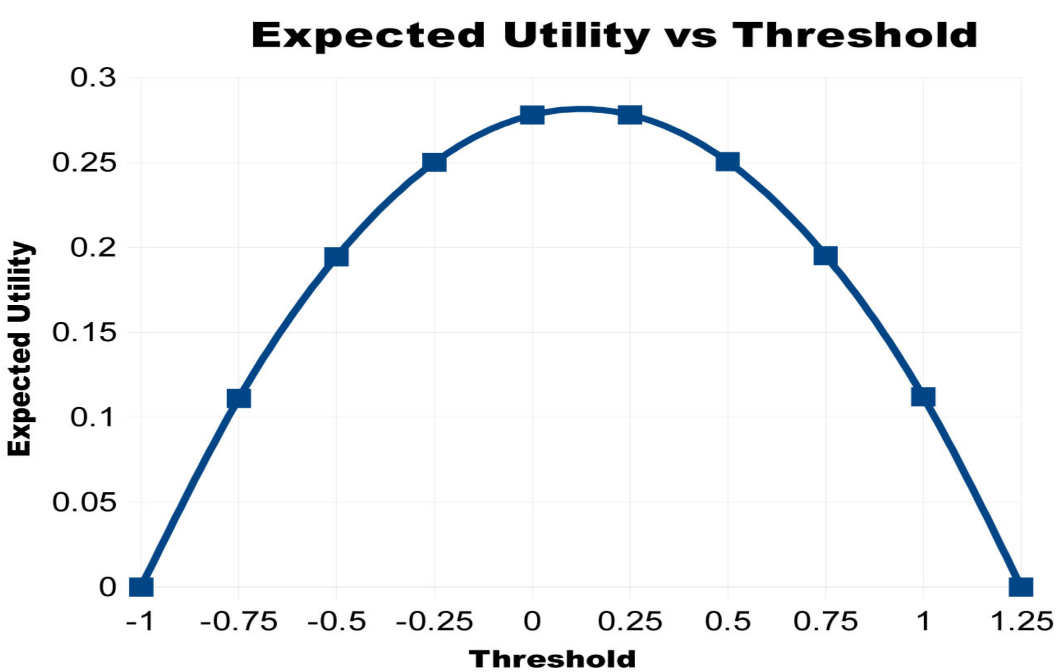
When the threshold is just over +1, $n_a = 0$, $u_a = 0$, p = 0. We define the value of $E(V_a)$ to be 0 at a utility of 1+ (2/n-1). In general, for n large, $E(V_a)$ can be made to be zero for a value of utility equal to 1 + Δ with Δ being arbitrarily small.

Let's do an example for the following normalized data set:

$u_i \in$ {-1,-3/4,-1/2,-1/4,0,1/4,1/2,3/4,1)

Threshold under	p	u_a/n_a	$E(V_a)$
-1	1	0	0
-3/4	8/9	(1)(1/8) = 1/8	1/9
-1/2	7/9	(7/4)(1/7) = 1/4	7/36
-1/4	6/9	(9/4)(1/6) = 3/8	1/4
0	5/9	(10/4)(1/5) = 1/2	5/18
+1/4	4/9	(10/4)(1/4) = 5/8	5/18
+1/2	3/9	(9/4)(1/3) = 3/4	1/4
+3/4	2/9	(7/4)(1/2) = 7/8	7/36
1	1/9	1	1/9
+5/4	0	0	0

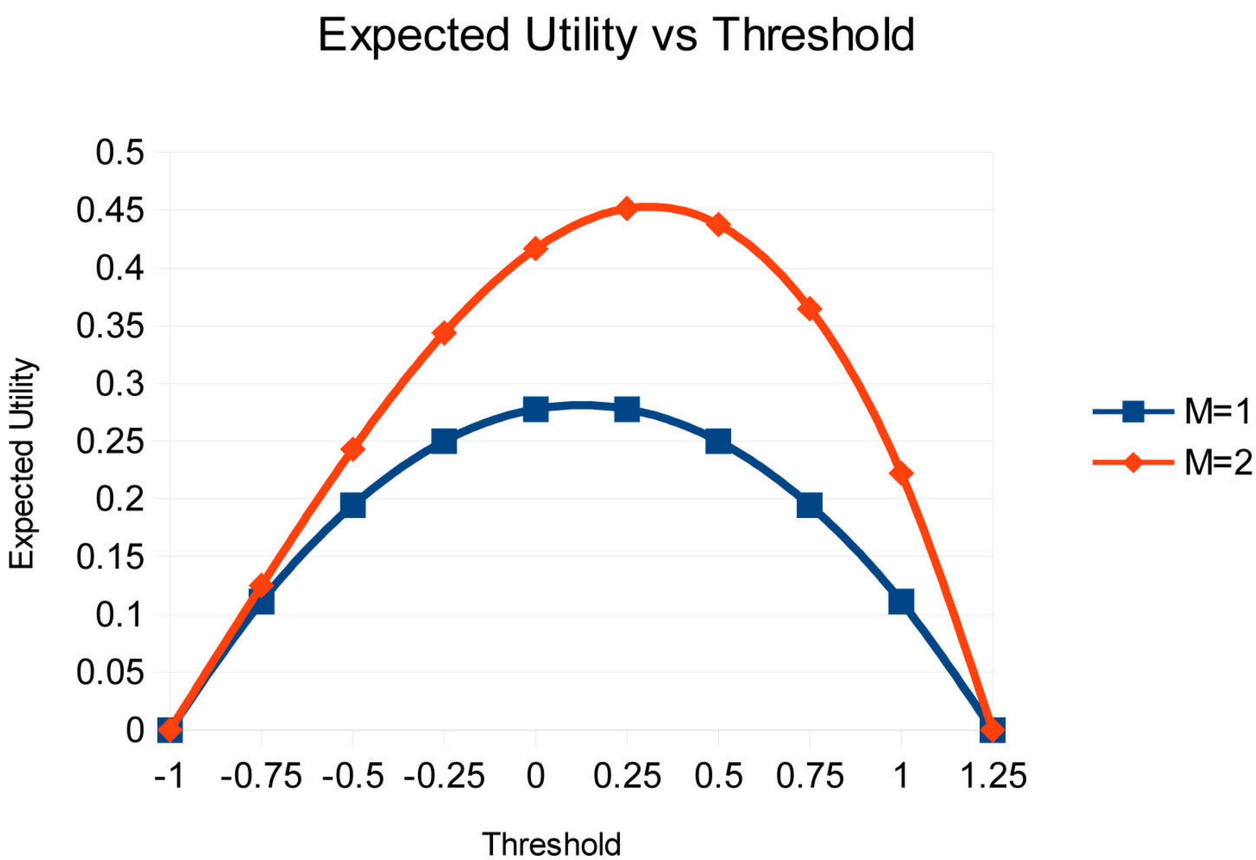
When there is one member in the winning set, expected social utility for an individual chooser is a maximum when the threshold is close to $u_i = 0$, $n_a = (n - 1)/2$. The maximum value of expected utility can be made to occur arbitrarily close to a threshold of zero by increasing n. The maximum value of expected utility can be made to occur arbitrarily close to a threshold of zero by increasing n. The graph is as follows:



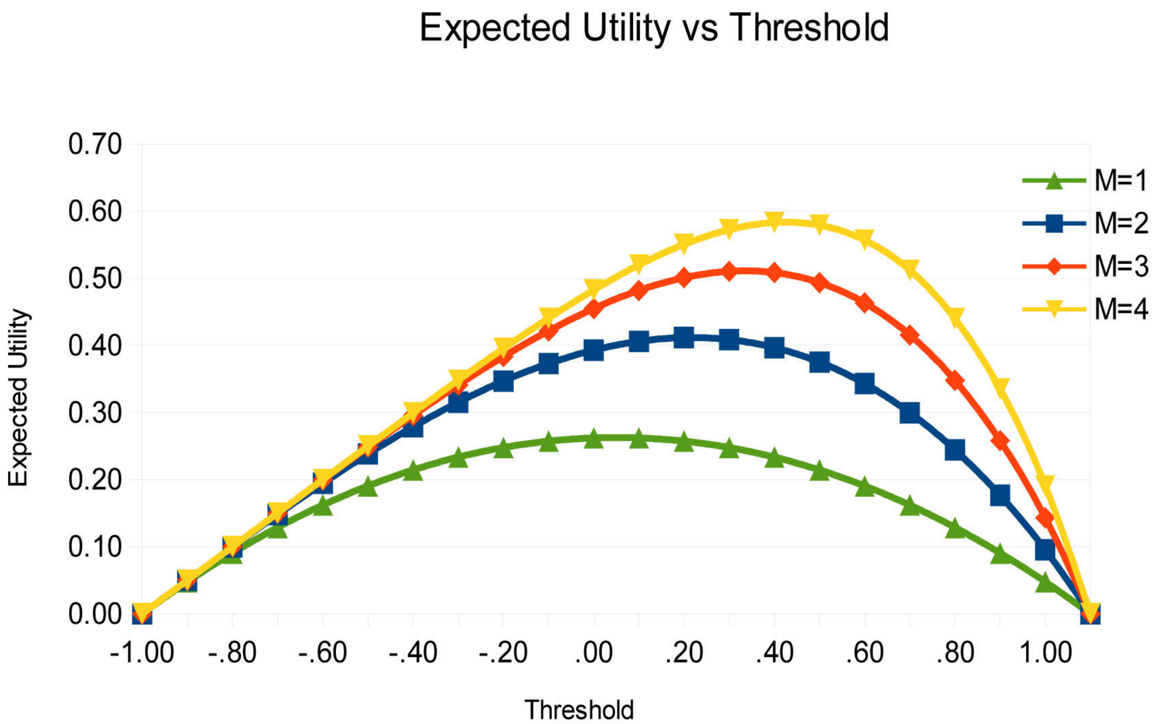
Appendix 2

We can see that the peak has shifted to the right and upwards indicating that the threshold for which expected average utility is maximum has shifted up towards greater utilities and the expected average utility at that threshold is greater.

As m increases, the individual chooser should derive increased utility or satisfaction from the winning set since one or more of their above threshold candidates are more likely to become part of the winning set W.



Appendix 3


Expected Utility vs Threshold
0.70
0.60
0.50
0.40
0.30
0.20
0.10
0.00
Expected Utility
-1.00
-.80
-.60
-.40
-.20
.00
.20
.40
.60
.80
1.00
Threshold
M=1
M=2
M=3
M=4

Appendix 4

Theorem: For the OTSC system, if a candidate drops out of an election after voting has occurred, the results of the election will not be changed for the other candidates.

Proof:

For some particular voter the expected value of above threshold utility is

$$\mathbf{E}(\mathbf{V}_a) = \mathbf{p}\left(\frac{\mathbf{1}}{\mathbf{n}_a}\right)\sum_{i=1}^{\mathbf{n}_a}\mathbf{u}_i$$

Assume candidate **j**, an above threshold candidate, drops out after votes are cast.

Let $\mathbf{p}_1$ be the value of **p** before the candidate drops out and $\mathbf{p}_2$ be the value after the drop out.

The expected value of above threshold utility after the drop out is

$$\mathbf{E}(\mathbf{V}_a) = \left(\frac{\mathbf{p}_2}{\mathbf{n}_a - \mathbf{1}}\right)(\mathbf{u}_1 + \mathbf{u}_2 + \dots + \mathbf{u}_{j-1} + \mathbf{u}_{j+1} + \dots + \mathbf{u}_{n_a})$$

Let $\mathbf{u}_1$ be the least above threshold value. Then the social choice may be changed if the optimal threshold is raised to exclude this value.

That would be true if

$$\frac{\mathbf{p}_2\left(\sum_{i=2,i\neq j}^{n_a}\mathbf{u}_i\right)}{\mathbf{n}_a - \mathbf{2}} > \frac{\mathbf{p}_1\left(\sum_{i=1,i\neq j}^{n_a}\mathbf{u}_i\right)}{\mathbf{n}_a - \mathbf{1}}$$

$$\left(\frac{p_2}{p_1}\right)\left(\sum_{i=2,i\neq j}^{n_a} u_i\right) > \left(\frac{n_a - 2}{n_a - 1}\right)\sum_{i=1,i\neq j}^{n_a} u_i$$

$p_2 < p_1$

$$\left(\sum_{i=2,i\neq j}^{n_a} u_i\right) > \left(\frac{n_a - 2}{n_a - 1}\right)\sum_{i=1,i\neq j}^{n_a} u_i$$

$$\left(\frac{n_a - 2}{n_a - 1}\right) < 1$$

So

$$\sum_{i=2,i\neq j}^{n_a} u_i > \sum_{i=1,i\neq j}^{n_a} u_i$$

But

$$\sum_{i=2,i\neq j}^{n_a} u_i \not> \sum_{i=1,i\neq j}^{n_a} u_i$$

and the assertion is proved by contradiction.

Therefore, the above threshold value of expected utility is not increased by raising the threshold if an above threshold candidate drops out. The optimal threshold remains the same.

We now prove that the above threshold value of expected utility is not increased by lowering the threshold if an above threshold candidate drops out. Proof is by contradiction.

Assume the optimal threshold is lowered to include the greatest value of $\mathbf{u_i}$ beneath the optimal threshold if an above threshold candidate $\mathbf{u_j}$ drops out. We renumber this utility $\mathbf{u_1}$. Let $\mathbf{p_1}$ be the value of $\mathbf{p}$ before the threshold is lowered and before $\mathbf{u_j}$ drops out corresponding to $\mathbf{n_a}$ above threshold values of $\mathbf{u_i}$ and $\mathbf{p_2}$ be the value of $\mathbf{p}$ corresponding to $\mathbf{n_a + 1}$ values of $\mathbf{u_i}$ before the threshold is lowered. Let $\mathbf{p_1'}$ be the value of $\mathbf{p}$ after the threshold is lowered corresponding to

$\mathbf{n_a - 1}$ above threshold values of $\mathbf{u_i}$ and $\mathbf{p_2'}$ be the value of $\mathbf{p}$ after the threshold is lowered corresponding to the $\mathbf{n_a}$ above threshold values of $\mathbf{u_i}$.

$$\mathbf{E(Va) = p\left(\frac{1}{n_a}\right)\sum_{i=1}^{n_a} u_i}$$

We know:

$$\frac{\mathbf{p_2}\left(\sum_{i=1}^{n_a+1} \mathbf{u_i}\right)}{\mathbf{n_a + 1}} < \frac{\mathbf{p_1}\left(\sum_{i=2}^{n_a+1} \mathbf{u_i}\right)}{\mathbf{n_a}}$$

by assumption.

We know that $\mathbf{p_2} > \mathbf{p_1}$ because the value of $\mathbf{p}$ is proportional to the number of above threshold utilities.

To prove:

$$\frac{\mathbf{p'_2}\left(\sum_{i=1, i\neq j}^{n_a+1} \mathbf{u_i}\right)}{\mathbf{n_a}} > \frac{\mathbf{p'_1}\left(\sum_{i=2, i\neq j}^{n_a+1} \mathbf{u_i}\right)}{\mathbf{n_a - 1}}$$

$$\left(\frac{\mathbf{p'_2}}{\mathbf{n_a}}\right)\left[\mathbf{u_1} + \left(\sum_{i=2, i\neq j}^{n_a+1} \mathbf{u_i}\right)\right] > \left(\frac{\mathbf{p'_1}}{\mathbf{n_a - 1}}\right)\left(\sum_{i=2, i\neq j}^{n_a+1} \mathbf{u_i}\right)$$

$$\mathbf{p'_2 = p_1}$$

$$\mathbf{p'_1 < p_1}$$

$$\mathbf{u}_1 + \left(\sum_{i=2, i\neq j}^{n_a+1} \mathbf{u}_i\right) > \left(\frac{\mathbf{p'}_1}{\mathbf{p}_1}\right)\left(\frac{\mathbf{n}_a}{\mathbf{n}_a - 1}\right)\left(\sum_{i=2, i\neq j}^{n_a+1} \mathbf{u}_i\right)$$

$$\mathbf{u}_1 + \left(\sum_{i=2, i\neq j}^{n_a+1} \mathbf{u}_i\right) > \left(\frac{\mathbf{n}_a}{\mathbf{n}_a - 1}\right)\left(\sum_{i=2, i\neq j}^{n_a+1} \mathbf{u}_i\right)$$

$$\mathbf{u}_1 > \left(\frac{\mathbf{n}_a}{\mathbf{n}_a - 1} - 1\right)\left(\sum_{i=2, i\neq j}^{n_a+1} \mathbf{u}_i\right)$$

$$\mathbf{u}_1 > \left(\frac{1}{\mathbf{n}_a - 1}\right)\left(\sum_{i=2, i\neq j}^{n_a+1} \mathbf{u}_i\right)$$

But

$$\mathbf{u}_1 \not> \left(\frac{1}{\mathbf{n}_a - 1}\right)\left(\sum_{i=2, i\neq j}^{n_a+1} \mathbf{u}_i\right)$$

The right hand side of the above equation represents the average value of utilities greater than $\mathbf{u}_1$. This is a contradiction since $\mathbf{u}_1$ is not greater than the average of utilities greater than $\mathbf{u}_1$ since each utility greater than $\mathbf{u}_1$ is greater than $\mathbf{u}_1$.

Therefore, the optimal threshold is not lowered if an above threshold candidate drops out after the election takes place.

Assume candidate j, a below threshold candidate, drops out after votes are cast. Assume that this would lower the optimal threshold. Let's renumber $\mathbf{u}_1$ as the greatest utility below the optimal threshold. Let $\mathbf{p}_1$ be the value of $\mathbf{p}$ before the threshold is lowered and before $\mathbf{u}_j$ drops out corresponding to $\mathbf{n}_a$ above threshold values of $\mathbf{u_i}$ and $\mathbf{p_2}$ be the value of $\mathbf{p}$ corresponding to $\mathbf{n_a + 1}$ values of $\mathbf{u_i}$ if the threshold were to be under $\mathbf{u_1}$.

We know:

$$\frac{\mathbf{p}_2\left(\sum_{i=1}^{n_a+1}\mathbf{u}_i\right)}{\mathbf{n}_a+1} < \frac{\mathbf{p}_1\left(\sum_{i=2}^{n_a+1}\mathbf{u}_i\right)}{\mathbf{n}_a}$$

by definition so the assumption is false.

Assume candidate j, a below threshold candidate, drops out after votes are cast. Assume that this would raise the optimal threshold. We renumber so that $\mathbf{u_1}$ is the first utility above the optimal threshold.

We know:

$$\frac{\mathbf{p}_2\left(\sum_{i=2}^{n_a}\mathbf{u}_i\right)}{\mathbf{n}_a-1} < \frac{\mathbf{p}_1\left(\sum_{i=1}^{n_a}\mathbf{u}_i\right)}{\mathbf{n}_a}$$

by definition so the assumption is false.

Since the optimal threshold doesn't have to be recomputed if a candidate drops out after the votes are cast by the voters, Arrow's IIA condition is preserved and the vote tally remains the same as if the candidate had just been blotted out of the election results. If the candidate who dropped out was in the winning set, the candidate with the highest vote total who was not in the winning set would then be elevated to it.

Appendix 5

To prove: Given any arbitrary individual utility scale consisting of preference ratings as inputs, the social choice results, when processed by the OTSC, will be the same as they would be for any affine linear transformation of that scale.

For some particular voter the expected value of above threshold average utility is

$$\mathbf{E(V_a)=\sum_{i=1}^{n_a} p_i u_i}$$

Let's assume that the optimal threshold is just under p_2u_2 so that

$$\mathbf{E(V_a) = (p_2u_2 + \ldots + p_{na}u_{na})/(n_a-1)}$$

where $p_ju_j < p_{j+1}u_{j+1}$ for $1 \le j < n_a$. We perform a linear affine transformation of the form $f(x) = ax + b$ (a nd b integers) and assume that the optimal threshold will move down from just under p_2u_2 to just under p_1u_1 so that the above threshold average utility is now

$$\mathbf{(p_1u_1 + \ldots + p_{na}u_{na})/n_a}$$

We assume:

$$\frac{\sum_1^n (au_j + b)}{n} > \frac{\sum_2^n (au_j + b)}{n-1}$$

We know:

$$\frac{\sum_1^n (au_j + b)}{n} = \frac{\sum_1^n (au_j) + nb}{n} = \frac{a\sum_1^n (u_j)}{n} + b$$

$$\frac{\sum_2^n (au_j + b)}{n-1} = \frac{\sum_2^n (au_j) + (n-1)b}{n-1} = \frac{a\sum_2^n (u_j)}{n-1} + b$$

So is

$$\frac{\mathbf{a}\sum_{1}^{n}(\mathbf{u}_j)}{\mathbf{n}}+\mathbf{b}>\frac{\mathbf{a}\sum_{2}^{n}(\mathbf{u}_j)}{\mathbf{n}-\mathbf{1}}+\mathbf{b}$$

Subtracting b from both sides and dividing by a we have

$$\frac{\sum_{1}^{n}(\mathbf{u}_j)}{\mathbf{n}}>\frac{\sum_{2}^{n}(\mathbf{u}_j)}{\mathbf{n}-\mathbf{1}}$$

However, we know that

$$\frac{\sum_{1}^{n}(\mathbf{u}_j)}{\mathbf{n}}<\frac{\sum_{2}^{n}(\mathbf{u}_j)}{\mathbf{n}-\mathbf{1}}$$

because by definition the optimal threshold is placed just under the utility such that the average utility above threshold is a maximum.

So the assumption is not true.

Similarly, if the average above threshold utility is $(p_1u_1 +\ldots + p_{na}u_{na})/n_a$, we show that applying an affine linear transformation and assuming that the optimal threshold moves up to just under p_2u_2 is false.

Assume that

$$\frac{\sum_{2}^{n}(\mathbf{au}_j+\mathbf{b})}{\mathbf{n}-\mathbf{1}}>\frac{\sum_{1}^{n}(\mathbf{au}_j+\mathbf{b})}{\mathbf{n}}$$

Then

$$\frac{\sum_2^n (au_j + b)}{n-1} = \frac{\sum_2^n (au_j) + (n-1)b}{n-1} = \frac{a\sum_2^n (u_j)}{n-1} + b$$

and

$$\frac{\sum_1^n (au_j + b)}{n} = \frac{\sum_1^n (au_j) + nb}{n} = \frac{a\sum_1^n (u_j)}{n} + b$$

Therefore,

$$\frac{a\sum_2^n (u_j)}{n-1} + b > \frac{a\sum_1^n (u_j)}{n} + b$$

and

$$\frac{a\sum_2^n (u_j)}{n-1} > \frac{a\sum_1^n (u_j)}{n}$$

$$\frac{\sum_2^n (u_j)}{n-1} > \frac{\sum_1^n (u_j)}{n}$$

But we know that,

$$\frac{\sum_1^n (u_j)}{n} > \frac{\sum_2^n (u_j)}{n-1}$$

by definition of the optimal threshold and the assumption is false. QED.

Therefore, an affine linear transformation does not change the placement of the optimal threshold.

Appendix 6

Example of Optimal Threshold Social Choice System

Let's assume there is a figure skating competition with 9 contestants, enumerated 1 through 9, and 5 judges alphabetized as A, B, C, D and E. The judges have to rate each contestant based on their performances. The top 3 overall will proceed to the next level of competition so the winning set will contain 3 contestants.

The expected value of utility of the winning set for each judge is the following:

$$\mathbf{E}(\mathbf{V}_j) = \sum_{k=1}^{s} \left\{ \left[\frac{\binom{\mathbf{n}_a}{\mathbf{k}}\binom{\mathbf{n}-\mathbf{n}_a}{\mathbf{m}-\mathbf{k}}}{\binom{\mathbf{n}}{\mathbf{m}}} \right] \left[\frac{\mathbf{k}}{\mathbf{n}_a} \right] \left[\sum_{i=1}^{n_a} \mathbf{u}_{ij} \right] \right\}$$

n = number of contestants = 9.

n_a = number of contestants above threshold.

m = number in winning set = 3.

$s = n_a$ if $n_a \leq m$ and $s = m$ if $n_a > m$

k = number of above threshold candidates in winning set.

u_{ij} = utility of contestant i for judge j, a decimal value between −1 and +1.

V_j = a random variable representing the utility of the winning set for judge j.

$E(V_j)$ = expected value of the utility of the winning set for judge j.

Each judge inputs a utility rating for each contestant based on their performance as shown in Table 1. The following data was randomly generated for each judge and for each contestant.

Table 1

Judge	A	B	C	D	E
Contestant					
1	-0.56	0.57	-0.18	-0.07	0.12
2	-0.04	0.11	-0.49	0.41	0.34
3	0.49	0.82	-0.21	-0.57	0.82
4	0.77	0.54	0.33	-0.06	0.21
5	-0.03	0.81	0.04	-0.25	-0.88
6	-0.29	0.45	-0.05	-0.93	-0.55
7	-0.43	-0.78	-0.31	-0.62	-0.16
8	-0.37	-0.12	0.14	0.49	-0.67
9	0.38	-0.05	-0.75	-0.81	-0.65

Table 2 shows the contestants listed from highest rated to lowest rated for each judge. Optimal thresholds were computed and are underlined.

Table 2

Judge A		Judge B		Judge C		Judge D		Judge E	
Contestant	Rating	Contestant	Rating	Contestant	Rating	Contestant	Rating	Contestant	Rating
4	0.77	3	0.82	4	0.33	8	0.49	3	0.82
3	0.49	5	0.81	8	0.14	2	0.41	2	0.34
5	-0.03	1	0.57	5	0.04	4	-0.06	4	0.21
2	-0.04	4	0.54	6	-0.05	1	-0.07	1	0.12
6	-0.29	6	0.45	1	-0.18	5	-0.25	7	-0.16
8	-0.37	2	0.11	3	-0.21	3	-0.57	6	-0.55
9	-0.38	9	-0.05	7	-0.31	7	-0.62	9	-0.65
7	-0.43	8	-0.12	2	-0.49	9	-0.81	8	-0.67
1	-0.56	7	-0.78	9	-0.75	6	-0.93	5	-0.88

Table 3 shows the scores for each contestant and judge after all scores above threshold have been raised to +1 and all scores below threshold lowered to -1.

<u>Table 3</u>

Judge A		Judge B		Judge C		Judge D		Judge E	
Contestant	Rating	Contestant	Rating	Contestant	Rating	Contestant	Rating	Contestant	Rating
4	1	3	1	4	1	8	1	3	1
3	<u>1</u>	5	1	8	1	2	1	2	1
5	-1	1	1	5	<u>1</u>	4	<u>1</u>	4	1
2	-1	4	1	6	-1	1	-1	1	<u>1</u>
6	-1	6	1	1	-1	5	-1	7	-1
8	-1	2	<u>1</u>	3	-1	3	-1	6	-1
9	-1	9	-1	7	-1	7	-1	9	-1
7	-1	8	-1	2	-1	9	-1	8	-1
1	-1	7	-1	9	-1	6	-1	5	-1

Table 4 shows the scores for each contestant.

<u>Table 4</u>

Contestant	Totals
1	-1
2	1
3	1
4	3
5	-1
6	-3
7	-5
8	-1
9	-5

So the winning set is comprised of contestants 2, 3 and 4. The following are the utilities computed for each judge:

Judge A - 1.22
Judge B - 1.47
Judge C - 0.37
Judge D - 0.22
Judge E - 1.37
Social Utility - 4.65

As a comparison the following table shows the Borda count for each judge and each contestant:

Table 5

Contestant	Borda Count Judge A	Borda Count Judge B	Borda Count Judge C	Borda Count Judge D	Borda Count Judge E
1	0	6	4	5	5
2	5	3	1	7	7
3	7	8	3	3	8
4	8	5	8	6	6
5	6	7	6	4	0
6	4	4	5	0	3
7	1	0	2	2	4
8	3	1	7	8	1
9	2	2	0	1	2

Following are the totals for each contestant:

Contestant	Totals	Order
1	20	4 tie
2	23	3 tie
3	29	2
4	33	1
5	23	3 tie
6	16	5
7	9	6
8	20	4 tie
9	7	7

The winning set is now comprised of contestants 3, 4 and a tie between contestants 2 and 5.

Following shows a table comprised of Arrovian pairwise comparisons illustrating the intransitivity among contestants 7, 8 and 9:

Table 6

Contestant	1	2	3	4	5	6	7	8	9
1		2>1	3>1	4>1	5=1	6<1	7<1	8=1	9<1
2			3=2	4>2	5<2	6<2	7<2	8<2	9<2
3				4>3	5<3	6<3	7<3	8<3	9<3
4					5<4	6<4	7<4	8<4	9<4
5						6<5	7<5	8=5	9<5
6							7<6	8>6	9<6
7								8<7	9=7
8									9<8
9									

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